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A TABLE OF ELLIPTIC INTEGRALS: TWO QUADRATIC FACTORS

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ABSTRACT. Thirteen integrands that are rational except for the square root of a quartic polynomial with two pairs of conjugate complex zeros are integrated in terms of R-functions of real variables. In contrast with previous tables, the formulas hold for all real intervals of integration for which the integrals exist (possibly as Cauchy principal values). This is achieved by using Landen's transformation and the duplication theorem. In an appendix, an elliptic integral of the third kind with a restricted complex parameter is transformed to make the parameter real. Also, a degenerate integral of the first kind is separated into real and imaginary parts.

1. INTRODUCTION

This paper treats integrands that are rational except for the square root of a quartic polynomial with two pairs of conjugate complex zeros. Integrals of the form

$$[p] = [p_1, \dots, p_5] = \int_y^x \prod_{i=1}^{5} (a_i + b_i t)^{p_i/2} \, dt, \tag{1.1}$$

where $p_1, \dots, p_4$ are odd integers and p_5 is even, are treated in [4, 5] if all quantities are real. Reference [8] deals with cases where $p_2 = p_3$ and $a_3 + b_3 t$ is the complex conjugate of $a_2 + b_2 t$. Here we assume further that $p_1 = p_4$ and $a_4 + b_4 t$ is the complex conjugate of $a_1 + b_1 t$. That is, we consider

$$[p_1, p_2, p_2, p_1, p_5] = \int_y^x \prod_{i=1}^{2} (f_i + g_i t + h_i t^2)^{p_i/2} (a_5 + b_5 t)^{p_5/2} \, dt, \tag{1.2}$$

where all quantities are real, $x > y$, $f_i + g_i t + h_i t^2 > 0$ for all real t, p_1 and p_2 are odd integers, and p_5 is even. We retain the redundant notation on the left side of (1.2), omitting p_5 if it is 0, for consistency with [5, 8]. Section 2 contains the 11 cases (apart from exchange of p_1 and p_2) with $2|p_1| + 2|p_2| + |p_5| \leq 8$ and $2p_1 + 2p_2 + p_5 \leq 0$, as well as $[1, 1, 1, 1, -2]$ and $[1, 1, 1, 1]$. The formulas hold for all x and y for which the integral exists (possibly as a Cauchy principal value if $p_5 = -2$).

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In Byrd and Friedman's table [1, §267] such integrals are listed only with a lower limit that depends on the parameters in the integrand, and a restriction on the upper limit is added in [9, 3.145(4)] so that ϕ in Legendre's $F(\phi, k)$ is between 0 and $\pi/2$. Also, there is an ambiguity of sign; e.g., [1, 267.00] with $a_1 = a_2 = \sqrt{2}$ is correct if $b_2 = -b_1 = 1/2$ and $y = 1$ but incorrect if $b_1 = -b_2 = 1/2$ and $y = 2$ unless g_1 is taken to be the negative square root of g_1^2.

The integrals (1.2) are expressed in terms of four R-functions:

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\,dt, \tag{1.3}$$

$$R_J(x, y, z, w) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+w)^{-1}\,dt, \tag{1.4}$$

and two special cases,

$$R_D(x, y, z) = R_J(x, y, z, z)$$

and

$$R_C(x, y) = R_F(x, y, y) = \frac{1}{2}\int_0^\infty (t+x)^{-1/2}(t+y)^{-1}\,dt. \tag{1.5}$$

The functions R_F, R_D, and R_J respectively replace Legendre's elliptic integrals of the first, second, and third kinds, while R_C, which requires special attention in this paper, includes the inverse circular (if $0 \le x < y$) and inverse hyperbolic (if $0 < y < x$) functions. Fortran codes for numerical computation of all four functions are listed in the Supplements to [4, 5] and are available in several major software libraries.

In [8] a Landen transformation was used to change the first two variables of R_F, R_D, and R_J from complex to real numbers; the remaining variables, including those of R_C, were never complex. In the present paper the complex variables are the parameter (the fourth variable) of R_J and both variables of R_C. However, a Landen transformation of R_F and R_D is used in §3 to eliminate a restriction on the interval of integration that arose in [3] because of a branch point. In §4 the complex parameter of R_J is made real, not by a direct Landen transformation but by an inverse Landen transformation followed by the duplication theorem (see Appendix A), a combination that also takes care of the branch-point problem. The function R_C with complex variables is separated into real and imaginary parts in Appendix B, and the imaginary part cancels another R_C that comes from the inverse Landen transformation of R_J.

The formulas of [5, 8] made it unnecessary to do any further work with recurrence relations, although conversion to notation appropriate for this paper sometimes entailed tedious algebra. The integrals I_2, I_3, and I_3' used previously are now complex, but the eventual cancellation of imaginary terms provided a partial check. The 13 integral formulas in §2 were checked by numerical integration; some details of the checks are given in §5. The variables of R_J and R_C are nonnegative, even when the three integrals with $p_5 = -2$ have their Cauchy principal values.

2. Table of integrals

We assume $x > y$ and $f_i + g_i t + h_i t^2 > 0$ for $i = 1, 2$ and all real t. Some relations useful for numerical checks are included among the following definitions:

(2.1) $$\xi_i = (f_i + g_i x + h_i x^2)^{1/2}, \qquad \eta_i = (f_i + g_i y + h_i y^2)^{1/2},$$

(2.2) $$\xi_1' = (g_1 + 2h_1 x)/2\xi_1, \qquad \eta_1' = (g_1 + 2h_1 y)/2\eta_1,$$

(2.3) $$B = \xi_1'\xi_2 - \eta_1'\eta_2, \qquad E = \xi_1'\xi_1^2\xi_2 - \eta_1'\eta_1^2\eta_2,$$

(2.4) $$\theta_i = 2f_i + g_i(x+y) + 2h_i xy = \xi_i^2 + \eta_i^2 - h_i(x-y)^2,$$

(2.5) $$\zeta_i = (2\xi_i\eta_i + \theta_i)^{1/2} = [(\xi_i + \eta_i)^2 - h_i(x-y)^2]^{1/2},$$

(2.6) $$U = (\xi_1\eta_2 + \eta_1\xi_2)/(x-y), \qquad M = \zeta_1\zeta_2/(x-y),$$

(2.7) $$\delta_{ij} = (2f_ih_j + 2f_jh_i - g_ig_j)^{1/2}, \qquad \Delta = (\delta_{12}^4 - \delta_{11}^2\delta_{22}^2)^{1/2},$$

(2.8) $$\Delta_\pm = \delta_{12}^2 \pm \Delta, \quad L_\pm^2 = M^2 + \Delta_\pm, \quad L_+L_- = 2MU,$$

(2.9) $$G = 2\Delta\Delta_+ R_D(M^2, L_-^2, L_+^2)/3 + \Delta/2U + (\delta_{12}^2\theta_1 - \delta_{11}^2\theta_2)/4\xi_1\eta_1 U,$$

(2.10) $$R_F = R_F(M^2, L_-^2, L_+^2), \qquad \Sigma = G - \Delta_+ R_F + B.$$

For integrals with $p_5 \neq 0$ we also define

(2.11) $$\alpha_{i5} = 2f_ib_5 - g_ia_5, \qquad \beta_{i5} = g_ib_5 - 2h_ia_5,$$

(2.12) $$\gamma_i = (\alpha_{i5}b_5 - \beta_{i5}a_5)/2 = f_ib_5^2 - g_ia_5b_5 + h_ia_5^2 > 0,$$

(2.13) $$\Lambda = \delta_{11}^2\gamma_2/\gamma_1, \qquad \Omega^2 = M^2 + \Lambda,$$

(2.14) $$\begin{aligned}\psi &= (\alpha_{15}\beta_{25} - \alpha_{25}\beta_{15})/2 \\ &= (g_1h_2 - g_2h_1)a_5^2 - 2(f_1h_2 - f_2h_1)a_5b_5 + (f_1g_2 - f_2g_1)b_5^2, \\ \psi^2 &= -\delta_{11}^2\gamma_2^2 + 2\delta_{12}^2\gamma_1\gamma_2 - \delta_{22}^2\gamma_1^2 = \gamma_1\gamma_2(\Delta_+ - \Lambda)(\Lambda - \Delta_-)/\Lambda,\end{aligned}$$

(2.15) $$\xi_5 = a_5 + b_5x, \qquad \eta_5 = a_5 + b_5y,$$

(2.16) $$A(p_1, p_2, p_2, p_1, p_5) = \xi_1^{p_1}\xi_2^{p_2}\xi_5^{p_5/2} - \eta_1^{p_1}\eta_2^{p_2}\eta_5^{p_5/2},$$

(2.17) $$\begin{aligned}X &= [\xi_5(\alpha_{15} + \beta_{15}y)\eta_2/\eta_1 + \eta_5(\alpha_{15} + \beta_{15}x)\xi_2/\xi_1]/2(x-y) \\ &= \xi_5\eta_5[\theta_1 A(-1, 1, 1, -1)/2 - \xi_5\eta_5 A(1, 1, 1, 1, -4)]/(x-y)^2,\end{aligned}$$

(2.18) $$S = (M^2 + \delta_{12}^2)/2 - U^2 = (\xi_1\eta_1\theta_2 + \xi_2\eta_2\theta_1)/(x-y)^2,$$

(2.19) $$\mu = \gamma_1\xi_5\eta_5/\xi_1\eta_1, \quad T = \mu S + 2\gamma_1\gamma_2, \quad V^2 = \mu^2(S^2 + \Lambda U^2),$$

(2.20) $$a = S\Omega^2/U + 2\Lambda U, \qquad b^2 = (S^2/U^2 + \Lambda)\Omega^4,$$

(2.21) $$a^2 = b^2 + \Lambda^2\psi^2/\gamma_1\gamma_2 = b^2 + \Lambda(\Delta_+ - \Lambda)(\Lambda - \Delta_-),$$

(2.22) $$\begin{aligned}H = {} & \delta_{11}^2\psi[R_J(M^2, L_-^2, L_+^2, \Omega^2)/3 + R_C(a^2, b^2)/2]/\gamma_1^2 \\ & - XR_C(T^2, V^2).\end{aligned}$$

We shall want some of the quantities above when $a_5 = 1$ and $b_5 = 0$. These will be labeled by a subscript 0:

(2.23) $$\Lambda_0 = \delta_{11}^2h_2/h_1, \qquad \Omega_0^2 = M^2 + \Lambda_0,$$

$$\begin{aligned}\psi_0 &= g_1h_2 - g_2h_1\,,\\ \psi_0^2 &= -\delta_{11}^2h_2^2 + 2\delta_{12}^2h_1h_2 - \delta_{22}^2h_1^2 = h_1h_2(\Delta_+ - \Lambda_0)(\Lambda_0 - \Delta_-)/\Lambda_0\,,\end{aligned} \tag{2.24}$$

$$\begin{aligned}X_0 &= -(\xi_1'\xi_2 + \eta_1'\eta_2)/(x-y)\\ &= [\theta_1A(-1\,,1\,,1\,,-1)/2 - A(1\,,1\,,1\,,1)]/(x-y)^2\,,\end{aligned} \tag{2.25}$$

$$\mu_0 = h_1/\xi_1\eta_1\,,\quad T_0 = \mu_0S + 2h_1h_2\,,\quad V_0^2 = \mu_0^2(S^2 + \Lambda_0U^2)\,, \tag{2.26}$$

$$a_0 = S\Omega_0^2/U + 2\Lambda_0U\,,\qquad b_0^2 = (S^2/U^2 + \Lambda_0)\Omega_0^4\,, \tag{2.27}$$

$$a_0^2 = b_0^2 + \Lambda_0^2\psi_0^2/h_1h_2 = b_0^2 + \Lambda_0(\Delta_+ - \Lambda_0)(\Lambda_0 - \Delta_-)\,, \tag{2.28}$$

$$\begin{aligned}H_0 = \delta_{11}^2\psi_0[R_J(M^2\,,L_-^2\,,L_+^2\,,\Omega_0^2)/3 + R_C(a_0^2\,,b_0^2)/2]/h_1^2\\ - X_0R_C(T_0^2\,,V_0^2)\,.\end{aligned} \tag{2.29}$$

If the interval of integration is infinite, convergent integrals (with $2p_1+2p_2+p_5 \le -4$) do not involve H_0. If $x \to +\infty$ and y is finite, we find (for $i = 1\,,2$) that

$$\xi_i \sim h_i^{1/2}x\,,\qquad \theta_i \sim (g_i + 2h_iy)x\,, \tag{2.30}$$

$$U = h_1^{1/2}\eta_2 + h_2^{1/2}\eta_1\,,\qquad M^2 = \prod_{i=1}^{2}(2h_i^{1/2}\eta_i + g_i + 2h_iy)\,. \tag{2.31}$$

If $y \to -\infty$ and x is finite, then (for $i = 1\,,2$)

$$\eta_i \sim h_i^{1/2}|y|\,,\qquad \theta_i \sim (g_i + 2h_ix)y\,, \tag{2.32}$$

$$U = h_1^{1/2}\xi_2 + h_2^{1/2}\xi_1\,,\qquad M^2 = \prod_{i=1}^{2}(2h_i^{1/2}\xi_i - g_i - 2h_ix)\,. \tag{2.33}$$

If $x = -y \to +\infty$, then (for $i = 1\,,2$)

$$\begin{aligned}\xi_i \sim \eta_i \sim h_i^{1/2}x\,,\quad \theta_i \sim -2h_ix^2\,,\quad \zeta_i = \delta_{ii}/h_i^{1/2}\,,\\ 1/U = M = R_C(a^2\,,b^2) = XR_C(T^2\,,V^2) = 0\,.\end{aligned} \tag{2.34}$$

In all three of the limiting cases an identity useful for (2.41) is

$$\begin{aligned}&B - b_5A(1\,,1\,,1\,,1\,,-2)\\ &\quad = (\alpha_{15} + \beta_{15}y)\eta_2/2\eta_1\eta_5 - (\alpha_{15} + \beta_{15}x)\xi_2/2\xi_1\xi_5\,.\end{aligned}$$

Aside from interchange of p_1 and p_2, there are 11 integrals

$$[p_1\,,p_2\,,p_2\,,p_1\,,p_5] = \int_y^x \prod_{i=1}^{2}(f_i + g_it + h_it^2)^{p_i/2}(a_5 + b_5t)^{p_5/2}\,dt \tag{2.35}$$

with odd integers $p_1\,,p_2$ and even integer p_5 such that $2|p_1| + 2|p_2| + |p_5| \le 8$ and $2p_1+2p_2+p_5 \le 0$. We shall include also $[1\,,1\,,1\,,1\,,-2]$ and $[1\,,1\,,1\,,1]$. The integral of the first kind is

$$[-1\,,-1\,,-1\,,-1] = 4R_F\,, \tag{2.36}$$

and the next two integrals are of the second kind:

$$[-3\,,1\,,1\,,-3] = 4(-G + \Delta_+R_F)/\delta_{11}^2\,, \tag{2.37}$$

$$(2.38)\qquad [-3, -1, -1, -3] = 8h_1[(\Lambda_0 - \delta_{12}^2)G/\Delta - (\Lambda_0 - \Delta_+)R_F]/\delta_{11}^2\Delta - 4\psi_0 A(-1, 1, 1, -1)/\Delta^2.$$

Like the three preceding integrals, three integrals of the third kind with $2p_1 + 2p_2 + p_5 \leq -4$ are not restricted to finite intervals of integration. They involve H but not H_0:

$$(2.39)\qquad [-1, -1, -1, -1, -2] = -2(b_5 H + \beta_{15} R_F/\gamma_1),$$

$$(2.40)\qquad \begin{aligned}&[1, -1, -1, 1, -4]\\ &= [\psi H + G + (\Lambda - \Delta_+)R_F]/\gamma_2\\ &\quad - [\beta_{15}A(-1, 1, 1, -1) + 2\gamma_1 A(-1, 1, 1, -1, -2)]/2b_5\gamma_2,\end{aligned}$$

$$(2.41)\qquad [-1, -1, -1, -1, -4] = b_5(\beta_{15}/\gamma_1 + \beta_{25}/\gamma_2)H + \beta_{15}^2 R_F/\gamma_1^2 + b_5^2[\Sigma - b_5 A(1, 1, 1, 1, -2)]/\gamma_1\gamma_2.$$

Seven integrals of the third kind have $2p_1 + 2p_2 + p_5 \geq -2$ and exist only for finite intervals of integration. Four of them with $p_5 \geq 0$ involve H_0 but not H:

$$(2.42)\qquad [-1, -1, -1, -1, 2] = 2b_5 H_0 - 2\beta_{15} R_F/h_1,$$

$$(2.43)\qquad [1, -1, -1, 1] = (\psi_0 H_0 + \Sigma + \Lambda_0 R_F)/h_2,$$

$$(2.44)\qquad [-1, -1, -1, -1, 4] = -b_5(\beta_{15}/h_1 + \beta_{25}/h_2)H_0 + b_5^2\Sigma/h_1h_2 + \beta_{15}^2 R_F/h_1^2,$$

$$(2.45)\qquad \begin{aligned}[1, 1, 1, 1] &= (\delta_{22}^2/h_2^2 - \delta_{11}^2/h_1^2)[\psi_0 H_0 + (\Lambda_0 - \delta_{12}^2)R_F]/8\\ &\quad - (3\psi_0^2 - 4h_1h_2\delta_{12}^2)(\Sigma + \delta_{12}^2 R_F)/24h_1^2h_2^2\\ &\quad + [\Delta^2 R_F - \psi_0 A(1, 1, 1, 1)]/12h_1h_2 + E/3h_1.\end{aligned}$$

The final three integrals have $p_5 < 0$ and involve both H and H_0:

$$(2.46)\qquad [1, -1, -1, 1, -2] = 2(-\gamma_1 H + h_1 H_0)/b_5,$$

$$(2.47)\qquad \begin{aligned}[1, 1, 1, 1, -2] &= -2\gamma_1\gamma_2 H/b_5^3 + [(h_1\gamma_2 + h_2\gamma_1)/b_5^3 - \psi_0^2/4h_1h_2b_5]H_0\\ &\quad + (\beta_{15}/h_1 + \beta_{25}/h_2)(\Sigma + \Lambda_0 R_F)/4b_5^2\\ &\quad - \delta_{11}^2\psi_0 R_F/2h_1^2 b_5 + A(1, 1, 1, 1)/2b_5,\end{aligned}$$

$$(2.48)\qquad \begin{aligned}[1, 1, 1, 1, -4] &= [-(\gamma_1\beta_{25} + \gamma_2\beta_{15})H + (h_1\beta_{25} + h_2\beta_{15})H_0]/b_5^3\\ &\quad + [2\Sigma + (\Lambda + \Lambda_0)R_F)]/b_5^2 - A(1, 1, 1, 1, -2)/b_5.\end{aligned}$$

3. Integrals of the first and second kinds

In this section we derive (2.36), (2.37), and (2.38). By [5, (2.13), (2.17)],

$$(3.1)\qquad \begin{aligned}I_1 &= \int_y^x \prod_{i=1}^4 (a_i + b_i t)^{-1/2}\,dt = 2R_F(U_{12}^2, U_{13}^2, U_{14}^2),\\ (x-y)U_{ij} &= X_iX_jY_kY_m + Y_iY_jX_kX_m,\\ X_i &= (a_i + b_i x)^{1/2}, \qquad Y_i = (a_i + b_i y)^{1/2},\end{aligned}$$

where i, j, k, m is any permutation of 1, 2, 3, 4. Let

$$(3.2)\qquad \begin{aligned} (a_1+b_1t)(a_4+b_4t) &= f_1+g_1t+h_1t^2>0, \qquad -\infty<t<\infty,\\ (a_2+b_2t)(a_3+b_3t) &= f_2+g_2t+h_2t^2>0, \qquad -\infty<t<\infty. \end{aligned}$$

Then the biquadratic polynomial in the integrand of

$$(3.3)\qquad I_1=\int_y^x [(f_1+g_1t+h_1t^2)(f_2+g_2t+h_2t^2)]^{-1/2}\,dt$$

has two pairs of conjugate complex zeros. Equation (3.1) remains valid by the permanence of functional relations if the U_{ij} are in the open right half-plane. We shall see that $U_{14}>0$, and we may choose $U_{13}>0$. Although U_{12} is real, it may be negative, and (3.1) is then invalid if R_F is taken to have the principal value represented by the integral (1.3). The reason is that R_F, as a function of any one of its variables, has a branch point at the origin [2, §8.3]. When U_{12} describes a semicircle about 0 from the positive to the negative real axis, U_{12}^2 makes a complete circle and R_F returns to a different branch. Negative values of U_{12} may occur, as shown in [3, §4], when the quadrilateral whose vertices are the complex zeros has diagonals intersecting at an interior point of the interval of integration. The integral I_1 can then be expressed in terms of two standard integrals by breaking the interval of integration at the intersection. In the present paper we shall eliminate this complication by using Landen's transformation [7, (5.5)] to write

$$(3.4)\qquad \begin{gathered} I_1=4R_F(M^2, L_-^2, L_+^2), \qquad M=U_{12}+U_{13},\\ L_\pm=[(U_{14}+U_{12})(U_{14}+U_{13})]^{1/2}\pm[(U_{14}-U_{12})(U_{14}-U_{13})]^{1/2},\\ L_+L_-=2MU_{14}, \qquad L_\pm^2-M^2=[(U_{14}^2-U_{12}^2)^{1/2}\pm(U_{14}^2-U_{13}^2)^{1/2}]^2, \end{gathered}$$

where M, L_-, and L_+ will be proved nonnegative for every interval of integration. Alternatively, the duplication theorem could be used for the same purpose, but the resulting expressions are less simple.

In (3.2), since only f_i, g_i, and h_i are given, we may choose $b_1=b_4=h_1^{1/2}>0$, $b_2=b_3=h_2^{1/2}>0$, $\mathrm{Im}(a_1)>0$, and $\mathrm{Im}(a_2)>0$. If we assume x and y to be finite and take the principal branch of the square roots in (3.1), then X_1, Y_1, X_2, Y_2 lie in the open first quadrant of the complex plane, while their respective complex conjugates X_4, Y_4, X_3, Y_3 lie in the open fourth quadrant. It is clear that $U_{14}>0$ because both terms of $(x-y)U_{14}$ are strictly positive and we assume $x>y$. The same assumption, along with $\mathrm{Im}(a_1)>0$, shows that

$$\mathrm{Im}(X_1^2Y_4^2)=\mathrm{Im}[(a_1+b_1x)(a_4+b_4y)]=h_1^{1/2}(y-x)\mathrm{Im}(a_1)<0.$$

Thus, X_1Y_4 is in the open fourth quadrant if x and y are finite, and a similar argument shows that X_3Y_2 is in the open first quadrant. Hence the product $X_1X_3Y_2Y_4$ is in the open right half-plane, its real part is positive, and $U_{13}>0$. Because X_2Y_3 is the complex conjugate of X_3Y_2 and so is in the open fourth quadrant, $X_1X_2Y_3Y_4$ is in the open lower half-plane, and U_{12} may be positive or negative. However, $X_1Y_4+Y_1X_4$ and $X_2Y_3+Y_2X_3$ are both strictly positive, whence

$$(3.5)\qquad U_{12}+U_{13}=(X_1Y_4+Y_1X_4)(X_2Y_3+Y_2X_3)/(x-y)>0.$$

It follows from (3.1) that

$$U_{ik}^2 - U_{im}^2 = d_{ij}d_{km}, \qquad d_{ij} = a_i b_j - a_j b_i. \tag{3.6}$$

In particular, we see that

$$\begin{aligned} U_{14}^2 - U_{13}^2 &= d_{12}d_{43} = |d_{12}|^2 > 0, \\ U_{13}^2 - U_{12}^2 &= d_{14}d_{32} > 0, \\ U_{14}^2 - U_{12}^2 &= d_{13}d_{42} = |d_{13}|^2 > 0. \end{aligned} \tag{3.7}$$

Note that $d_{12} \neq 0$ because we exclude the degenerate case where $a_1 + b_1 t$ is proportional to $a_2 + b_2 t$ (whence $f_1 + g_1 t + h_1 t^2$ is proportional to $f_2 + g_2 t + h_2 t^2$, and the integral (3.3) is elementary). The second inequality holds because d_{14} is positive imaginary and d_{32} is negative imaginary. Finally, $d_{13} \neq 0$ because $\operatorname{Im} a_1 > 0$ and $\operatorname{Im} a_3 < 0$. We may now conclude, if x and y are finite, that

$$U_{14} > U_{13} > 0 \quad \text{and} \quad -U_{13} < U_{12} < U_{13}. \tag{3.8}$$

By (3.4) it follows that $M > 0$, $L_+ > L_- > 0$, and $L_-^2 - M^2 > 0$, whence

$$L_+ > L_- > M > 0. \tag{3.9}$$

Since $L_\pm^2 - M^2$ depends only on the quantities listed in (3.7), which are independent of x and y, both (3.9) and (3.8) are still valid if either x or y is infinite, but not both. If the interval of integration is the whole real line, then $U_{14} = U_{13} = -U_{12} = +\infty$ and $M = 0$, as we shall show later.

Again assuming x and y to be finite, we shall now express M and $L_\pm$ in terms of f_i, g_i, and h_i. Let

$$\begin{aligned} \xi_1 &= X_1X_4 = (f_1 + g_1 x + h_1 x^2)^{1/2}, & \xi_2 &= X_2X_3 = (f_2 + g_2 x + h_2 x^2)^{1/2}, \\ \eta_1 &= Y_1Y_4 = (f_1 + g_1 y + h_1 y^2)^{1/2}, & \eta_2 &= Y_2Y_3 = (f_2 + g_2 y + h_2 y^2)^{1/2}, \\ \zeta_1 &= X_1Y_4 + Y_1X_4 = 2\operatorname{Re}(X_1Y_4), & \zeta_2 &= X_2Y_3 + Y_2X_3 = 2\operatorname{Re}(X_2Y_3), \\ \theta_1 &= X_1^2Y_4^2 + Y_1^2X_4^2 = \zeta_1^2 - 2\xi_1\eta_1, & \theta_2 &= X_2^2Y_3^2 + Y_2^2X_3^2 = \zeta_2^2 - 2\xi_2\eta_2. \end{aligned} \tag{3.10}$$

Then ξ_i, η_i, and ζ_i are positive, but θ_i need not be. By (3.4) and (3.5) we see that

$$M = \zeta_1\zeta_2/(x-y), \qquad \zeta_i^2 = (\xi_i + \eta_i)^2 - h_i(x-y)^2, \tag{3.11}$$

where the second equation follows from

$$\begin{aligned} \zeta_1^2 = (X_1Y_4 + Y_1X_4)^2 &= (X_1X_4 + Y_1Y_4)^2 - (X_1^2 - Y_1^2)(X_4^2 - Y_4^2) \\ &= (\xi_1 + \eta_1)^2 - h_1(x-y)^2, \end{aligned} \tag{3.12}$$

and similarly for ζ_2^2.

If we define

$$\delta_{ij} = (2f_ih_j + 2f_jh_i - g_ig_j)^{1/2}, \tag{3.13}$$

then $\delta_{ii} > 0$ because $f_i + g_i t + h_i t^2 > 0$ for all real t. A stronger result than $\delta_{12} > 0$ will be given in (3.17). By (3.2) we have

$$a_1b_4 + a_4b_1 = g_1, \qquad (a_1b_4)(a_4b_1) = f_1h_1.$$

We solve these two equations for $a_1 b_4$ and use the assumption $\mathrm{Im}(a_1) > 0$ to find

$$a_1 b_4 = (g_1 + i\delta_{11})/2, \qquad d_{14} = a_1 b_4 - a_4 b_1 = i\delta_{11}. \tag{3.14}$$

A similar procedure yields

$$a_2 b_3 = (g_2 + i\delta_{22})/2, \qquad d_{23} = i\delta_{22}. \tag{3.15}$$

Thus we find

$$\begin{aligned} |d_{12}^2| &= d_{12} d_{43} = (a_1 b_2 - a_2 b_1)(a_4 b_3 - a_3 b_4) \\ &= f_1 h_2 + f_2 h_1 - 2\,\mathrm{Re}[(g_1 + i\delta_{11})(g_2 - i\delta_{22})/4] \\ &= (\delta_{12}^2 - \delta_{11}\delta_{22})/2. \end{aligned} \tag{3.16}$$

Since $d_{12} \neq 0$, except in the excluded degenerate case (cf. (3.7)), we have

$$\delta_{12}^2 > \delta_{11}\delta_{22} > 0. \tag{3.17}$$

A similar calculation leads to

$$|d_{13}^2| = (\delta_{12}^2 + \delta_{11}\delta_{22})/2. \tag{3.18}$$

We define

$$\Delta = (\delta_{12}^4 - \delta_{11}^2 \delta_{22}^2)^{1/2}, \qquad \Delta_\pm = \delta_{12}^2 \pm \Delta > 0, \tag{3.19}$$

and use (3.7) and the last equation of (3.4) to get

$$\begin{aligned} &U_{14}^2 - U_{12}^2 = (\delta_{12}^2 + \delta_{11}\delta_{22})/2, \qquad && U_{14}^2 - U_{13}^2 = (\delta_{12}^2 - \delta_{11}\delta_{22})/2, \\ &U_{13}^2 - U_{12}^2 = \delta_{11}\delta_{22}, && L_\pm^2 - M^2 = \Delta_\pm. \end{aligned} \tag{3.20}$$

The last equation and (3.11) allow calculation of M^2 and $L_\pm^2$.

If the interval of integration is infinite, we take the appropriate limit in (3.11) to find (2.31), (2.33), or (2.34). In the first two cases, M^2 is a product of two factors such as

$$2h_i^{1/2}\xi_i - g_i - 2h_i x = [(g_i + 2h_i x)^2 + \delta_{ii}^2]^{1/2} - (g_i + 2h_i x) > 0.$$

Hence, $M > 0$, except when the interval of integration is the whole real line.

An integral of the second kind used in previous parts of this table [5, (2.14), (2.17)] is

$$\begin{aligned} I_2 &= [1, -1, -1, -3] \\ &= 2 d_{12} d_{13} R_D(U_{12}^2, U_{13}^2, U_{14}^2)/3 + 2X_1 Y_1 / X_4 Y_4 U_{14}. \end{aligned} \tag{3.21}$$

(Since $p_1 \neq p_4$, this integral is now complex.) Putting $w = z$ in [7, (8.5), (5.5)] to obtain the Landen transformation of R_D, and using the notation in (3.4), we find

$$\begin{aligned} \Delta R_D(U_{12}^2, U_{13}^2, U_{14}^2) = 8\Delta_+ R_D(M^2, L_-^2, L_+^2) \\ - 12 R_F(M^2, L_-^2, L_+^2) + 6/U_{14}. \end{aligned} \tag{3.22}$$

Substituting in (3.21) and using the identities

$$\begin{aligned} &d_{12} d_{13} d_{24} d_{34} = |d_{12} d_{13}|^2 = \Delta^2/4, \\ &4 d_{24} d_{34} X_1^2 Y_1^2 = \delta_{12}^2 \theta_1 - \delta_{11}^2 \theta_2 - i2\delta_{11}\xi_1\eta_1 U_{14} A(-1, 1, 1, -1), \end{aligned} \tag{3.23}$$

we get

$$I_2 = 4d_{12}d_{13}[2G - 2\Delta R_F - i\delta_{11}A(-1\,,\,1\,,\,1\,,\,-1)]/\Delta^2\,, \tag{3.24}$$

$$G = 2\Delta\Delta_+ R_D(M^2\,,\,L_-^2\,,\,L_+^2)/3 + \Delta/2U + (\delta_{12}^2\theta_1 - \delta_{11}^2\theta_2)/4\xi_1\eta_1 U\,, \tag{3.25}$$

where we denote $R_F(M^2\,,\,L_-^2\,,\,L_+^2)$ by R_F and U_{14} by U for brevity.

The integrals $[-3\,,\,1\,,\,1\,,\,-3]$ and $[-3\,,\,-1\,,\,-1\,,\,-3]$ can now be obtained from [8, (2.24), (2.23)]. In the second case, we use the identity

$$\begin{aligned} 2d_{12}d_{13} &= [h_1\delta_{12}^2 - h_2\delta_{11}^2 + i(g_1h_2 - g_2h_1)\delta_{11}]/h_1 \\ &= \delta_{12}^2 - \Lambda_0 + i\psi_0\delta_{11}/h_1\,. \end{aligned} \tag{3.26}$$

4. Integrals of the third kind

We shall encounter $R_J(U_{12}^2\,,\,U_{13}^2\,,\,U_{14}^2\,,\,W^2)$, where the first three variables are real but W^2 is complex. The function can be changed by Landen transformation [8, (4.14)] into $R_J(M^2\,,\,L_-^2\,,\,L_+^2\,,\,W_+^2)$, but W_+^2 also is complex. Instead, an inverse Landen transformation followed by the duplication theorem leads to $R_J(M^2\,,\,L_-^2\,,\,L_+^2\,,\,\Omega^2)$ with real Ω^2. This combination of two transformations (see Appendix A) is equivalent to a direct Landen transformation for integrals of the first and second kinds but not the third kind.

In (A.8) we identify $(z_-\,,\,z_+\,,\,\alpha)$ with $(U_{12}\,,\,U_{13}\,,\,U_{14})$ and find from (A.9) and (3.4) that $(x^2+\lambda\,,\,y^2+\lambda\,,\,z^2+\lambda) = (L_-^2\,,\,L_+^2\,,\,M^2)$. Because of [5, (2.15), (2.9)], we put

$$w_+^2 = W^2 = U_{14}^2 - d_{12}d_{13}d_{45}/d_{15}\,. \tag{4.1}$$

Since d_{15} and d_{45} are complex conjugates, it follows that $|\alpha^2 - w_+^2|^2 = |d_{12}d_{13}|^2$. By (3.7) the condition (A.2) is satisfied, and so w_-^2 is the complex conjugate of w_+^2:

$$w_-^2 = U_{14}^2 - d_{43}d_{42}d_{15}/d_{45}\,. \tag{4.2}$$

We define

$$\Omega^2 = w^2 + \lambda\,, \qquad w = w_+w_-/U_{14}\,, \tag{4.3}$$

and find from (A.3), (3.7), and [4, (5.22)] that

$$\begin{aligned} \Omega^2 - M^2 &= w^2 - z^2 = w_+^2 + w_-^2 - z_+^2 - z_-^2 \\ &= (U_{14}^2 - U_{13}^2) + (U_{14}^2 - U_{12}^2) - d_{12}d_{13}d_{45}/d_{15} - d_{43}d_{42}d_{15}/d_{45} \\ &= (d_{12}d_{45} - d_{42}d_{15})(d_{43}d_{15} - d_{13}d_{45})/d_{15}d_{45} \\ &= |(d_{12}d_{45} - d_{42}d_{15})/d_{15}|^2 = |d_{14}d_{25}/d_{15}|^2\,. \end{aligned}$$

Defining

$$\gamma_i = |d_{i5}|^2 = f_ib_5^2 - g_ia_5b_5 + h_ia_5^2\,, \qquad i = 1\,,\,2\,, \tag{4.4}$$

we note that $\gamma_i > 0$ because $\operatorname{Im} d_{i5} \neq 0$. By (3.14) we have

$$\Omega^2 = M^2 + \Lambda\,, \quad w^2 = z^2 + \Lambda\,, \quad \Lambda = \delta_{11}^2\gamma_2/\gamma_1 > 0\,. \tag{4.5}$$

Since $\Omega^2 > 0$, we may choose $\Omega > 0$. Then it follows from (3.9), (A.5), (4.3), and (3.20) that

$$L_+ > \Omega > L_- > M \geq 0\,, \qquad \Delta_+ > \Lambda > \Delta_- > 0\,, \tag{4.6}$$

where $M = 0$ by (2.34) only if the interval of integration is the whole real line. It will be useful to define also (for $i = 1, 2$)

$$\alpha_{i5} = \partial\gamma_i/\partial b_5 = 2f_i b_5 - g_i a_5, \qquad \beta_{i5} = -\partial\gamma_i/\partial a_5 = g_i b_5 - 2h_i a_5, \tag{4.7}$$

whence

$$\gamma_i = (\alpha_{i5} b_5 - \beta_{i5} a_5)/2. \tag{4.8}$$

(The definitions (4.7) are equivalent to $\alpha_{15} = a_1 d_{45} + a_4 d_{15}$ and $\beta_{15} = b_1 d_{45} + b_4 d_{15}$, and similar relations for α_{25} and β_{25}.)

From (4.1) and (4.2) we can obtain a coefficient in (A.8):

$$\begin{aligned} w_+^2 - w_-^2 &= -d_{12}d_{13}d_{45}/d_{15} + d_{43}d_{42}d_{15}/d_{45} \\ &= -(2i/\gamma_1)\mathrm{Im}(d_{12}d_{13}d_{45}^2). \end{aligned}$$

It is straightforward to show by (3.6) and (3.2) that

$$d_{12}d_{45} = a_5(h_1 a_2 - a_1 b_4 b_2) + b_5(f_1 b_2 - a_4 b_1 a_2).$$

Replacing the subscript 2 by 3, multiplying $d_{12}d_{45}$ by $d_{13}d_{45}$, and taking the imaginary part with the help of (3.14), we find

$$w_+^2 - w_-^2 = -i\delta_{11}\psi/\gamma_1, \tag{4.9}$$

where

$$\begin{aligned} \psi &= (\alpha_{15}\beta_{25} - \alpha_{25}\beta_{15})/2 \\ &= a_5^2(g_1 h_2 - g_2 h_1) - 2a_5 b_5(f_1 h_2 - f_2 h_1) + b_5^2(f_1 g_2 - f_2 g_1). \end{aligned}$$

Incidentally, with the help of (A.3) and (3.19) we see that

$$\begin{aligned} (w_+^2 - w_-^2)^2 &= (x^2 - w^2)(y^2 - w^2) = (L_-^2 - \Omega^2)(L_+^2 - \Omega^2) \\ &= (\Delta_- - \Lambda)(\Delta_+ - \Lambda) = \Lambda^2 - 2\delta_{12}^2\Lambda + \delta_{11}^2\delta_{22}^2. \end{aligned} \tag{4.10}$$

Substituting Λ from (4.5) and comparing with (4.9), we get

$$0 > (w_+^2 - w_-^2)^2 = -\Lambda\psi^2/\gamma_1\gamma_2, \tag{4.11}$$

where

$$\psi^2 = -\delta_{11}^2\gamma_2^2 + 2\delta_{12}^2\gamma_1\gamma_2 - \delta_{22}^2\gamma_1^2 > 0,$$

a result that is tedious to obtain by squaring ψ. Although it provides a useful numerical check, the last equation does not determine the sign of ψ, which may be positive or negative.

Since $b = w(w^2 + \lambda) = w\Omega^2$ by (A.6), we can now write (A.8) as

$$\begin{aligned} &2(W^2 - U_{14}^2)R_J(U_{12}^2, U_{13}^2, U_{14}^2, W^2) \\ &\quad = -i(\delta_{11}\psi/\gamma_1)[2R_J(M^2, L_-^2, L_+^2, \Omega^2) + 3R_C(a^2, b^2)] \\ &\qquad + 6R_F(M^2, L_-^2, L_+^2) - 3R_C(z^2, w^2), \end{aligned} \tag{4.12}$$

where W^2 is given in (4.1) and

$$\begin{aligned} z &= U_{12}U_{13}/U_{14}, \qquad w^2 - z^2 = \Omega^2 - M^2 = \Lambda = \delta_{11}^2\gamma_2/\gamma_1, \\ b &= w\Omega^2, \qquad a^2 - b^2 = \Lambda^2\psi^2/\gamma_1\gamma_2 = \Lambda(\Delta_+ - \Lambda)(\Lambda - \Delta_-). \end{aligned} \tag{4.13}$$

An integral of the third kind used in previous parts of this table [5, (2.15), (2.17)] is

$$(4.14)\qquad \begin{aligned} I_3 &= [1, -1, -1, -1, -2] \\ &= 2d_{12}d_{13}d_{14}R_J(U_{12}^2, U_{13}^2, U_{14}^2, W^2)/3d_{15} + 2R_C(P^2, Q^2). \end{aligned}$$

Substituting (4.12) and using (4.1), (3.14), and (4.4), we find

$$(4.15)\qquad \begin{aligned} I_3/2d_{15} = &-\delta_{11}^2\psi[R_J(M^2, L_-^2, L_+^2, \Omega^2)/3 + R_C(a^2, b^2)/2]/\gamma_1^2 \\ &- i\delta_{11}R_F/\gamma_1 + i\delta_{11}R_C(z^2, w^2)/2\gamma_1 + R_C(P^2, Q^2)/d_{15}, \end{aligned}$$

where $R_F = R_F(M^2, L_-^2, L_+^2)$.

We shall separate the real and imaginary parts of the last term,

$$(4.16)\qquad R_C(P^2, Q^2)/d_{15} = R_C((d_{15}P)^2, (d_{15}Q)^2),$$

and find that the imaginary part cancels the next to last term. Putting $d_{15}P = X + iY$ and referring to (B.1) in Appendix B, we shall need X, Y, $X^2 + Y^2$, $|d_{15}Q|^4$, and

$$(4.17)\qquad c = d_{15}^2(Q^2 - P^2) = -d_{25}d_{35}d_{15}d_{45} = -\gamma_1\gamma_2,$$

where we have used [5, (2.5)] and (4.4). It follows from [5, (2.8)] and (3.10) that

$$(4.18)\qquad (x-y)(X+iY) = (x-y)d_{15}P = (\eta_5\xi_2/\xi_1)d_{15}X_4^2 + (\xi_5\eta_2/\eta_1)d_{15}Y_4^2,$$

where

$$\xi_5 = X_5^2 = a_5 + b_5x, \qquad \eta_5 = Y_5^2 = a_5 + b_5y.$$

Using (3.14) and (4.7), we find

$$(4.19)\qquad 2d_{15}X_4^2 = 2(a_1b_5 - a_5b_1)(a_4 + b_4x) = \alpha_{15} + \beta_{15}x + i\delta_{11}\xi_5,$$

and similarly for $d_{15}Y_4^2$. Substitution in (4.18) yields

$$(4.20)\qquad \begin{aligned} 2(x-y)X &= \eta_5(\alpha_{15} + \beta_{15}x)\xi_2/\xi_1 + \xi_5(\alpha_{15} + \beta_{15}y)\eta_2/\eta_1, \\ 2Y &= \delta_{11}\xi_5\eta_5(\xi_2/\xi_1 + \eta_2/\eta_1)/(x-y) = \delta_{11}\xi_5\eta_5U/\xi_1\eta_1. \end{aligned}$$

Instead of squaring X and Y, it is easier to get $X^2 + Y^2$ by calculating $|d_{15}P|^2 = \gamma_1|P|^2$ from [5, (2.8)]:

$$(4.21)\qquad \begin{aligned} (x-y)^2|P|^2 &= (\xi_5\eta_2)^2 + (\eta_5\xi_2)^2 + \xi_5\eta_5\xi_2\eta_2(X_4Y_1/X_1Y_4 + X_1Y_4/X_4Y_1) \\ &= (\xi_5\eta_2)^2 + (\eta_5\xi_2)^2 + \xi_5\eta_5\theta_1\xi_2\eta_2/\xi_1\eta_1. \end{aligned}$$

From $\gamma_2 = d_{25}d_{35}$ and $(x-y)d_{i5} = \xi_5Y_i^2 - \eta_5X_i^2$ it follows by (3.10) that

$$(4.22)\qquad (x-y)^2\gamma_2 = (\xi_5\eta_2)^2 + (\eta_5\xi_2)^2 - \xi_5\eta_5\theta_2,$$

and hence

$$(4.23)\qquad (x-y)^2|P|^2 = \xi_5\eta_5(\theta_2 + \theta_1\xi_2\eta_2/\xi_1\eta_1) + (x-y)^2\gamma_2.$$

Defining

$$(4.24)\qquad S = Uz = U_{12}U_{13},$$

we find

$$(4.25)\quad \begin{aligned}(x-y)^2 S &= (X_1X_2Y_3Y_4 + Y_1Y_2X_3X_4)(X_1X_3Y_2Y_4 + Y_1Y_3X_2X_4)\\ &= \xi_1\eta_1(X_2^2Y_3^2 + Y_2^2X_3^2) + \xi_2\eta_2(X_1^2Y_4^2 + Y_1^2X_4^2)\\ &= \xi_1\eta_1\theta_2 + \xi_2\eta_2\theta_1\end{aligned}$$

and thus

$$(4.26)\quad \begin{aligned}&|P|^2 = \xi_5\eta_5 S/\xi_1\eta_1 + \gamma_2,\\ &X^2 + Y^2 = \gamma_1\xi_5\eta_5 S/\xi_1\eta_1 + \gamma_1\gamma_2 = \mu S + \gamma_1\gamma_2 = T - \gamma_1\gamma_2,\\ &\mu = \gamma_1\xi_5\eta_5/\xi_1\eta_1, \qquad T = \mu S + 2\gamma_1\gamma_2.\end{aligned}$$

Finally, [5, (2.5)], (4.1), and (4.3) imply

$$(4.27)\quad \begin{aligned}|d_{15}Q|^4 = (\mu w_+ w_-)^2 = (\mu U w)^2 = \mu^2 U^2 (z^2 + \Lambda) = V^2,\\ V^2 = \mu^2(S^2 + U^2\Lambda),\end{aligned}$$

while

$$(4.28)\quad X^2 + Y^2 + c = \mu S + \gamma_1\gamma_2 - \gamma_1\gamma_2 = \mu U z.$$

From (B.1) we now have

$$(4.29)\quad \begin{aligned}&R_C(P^2, Q^2)/d_{15}\\ &\quad = X R_C(T^2, V^2) - i(\delta_{11}\mu U/2\gamma_1) R_C((\mu U z)^2, (\mu U w)^2))\\ &\quad = X R_C(T^2, V^2) - i(\delta_{11}/2\gamma_1) R_C(z^2, w^2).\end{aligned}$$

The last term cancels a term in (4.15) to yield

$$(4.30)\quad I_3 = -2d_{15}(H + i\delta_{11}R_F/\gamma_1),$$

where

$$H = \delta_{11}^2\psi[R_J(M^2, L_-^2, L_+^2, \Omega^2)/3 + R_C(a^2, b^2)/2]/\gamma_1^2 - X R_C(T^2, V^2).$$

Using a subscript 0 to label quantities in which we have put $a_5 = 1$ and $b_5 = 0$, we obtain from [5, (2.17)] also

$$(4.31)\quad I_3' = [1, -1, -1, -1] = 2b_1(H_0 + i\delta_{11}R_F/h_1),$$

where

$$H_0 = \delta_{11}^2\psi_0[R_J(M^2, L_-^2, L_+^2, \Omega_0^2)/3 + R_C(a_0^2, b_0^2)/2]/h_1^2 - X_0 R_C(T_0^2, V_0^2).$$

Since I_1, I_2, I_3, and I_3' have now been reduced to R-functions of real variables, the ten integrals of the third kind in §2 can be derived by substitution in the formulas of [5, 8] (the latter if the odd p's are not in decreasing order). Converting coefficients to the notation of this paper is straightforward but sometimes tedious. In addition to the recurrence relation [5, (4.8)], the following identities are useful:

$$(4.32)\quad 2d_{24}d_{34}X_1^2Y_1^2 = (\delta_{12}^2\theta_1 - \delta_{11}^2\theta_2)/2 - i\delta_{11}\xi_1\eta_1 U A(-1, 1, 1, -1),$$

$$(4.33)\quad h_1^{1/2}A(1, 1, 1, -1) = B + i\delta_{11}A(-1, 1, 1, -1)/2,$$

$$(4.34)\quad h_1^{1/2}A(3, 1, 1, 1) = E + i\delta_{11}A(1, 1, 1, 1)/2,$$

where B and E are defined in (2.3).

5. NUMERICAL CHECKS

The 13 integrals in §2 were checked numerically when $x = 2$, $y = -3$, $(f_1, g_1, h_1) = (2.7, -1.8, 0.9)$, $(f_2, g_2, h_2) = (2.0, 2.4, 0.8)$, and $(a_5, b_5) = (1.1, -0.4)$. (The zeros of the quadratic polynomials are $1 \pm i\sqrt{2}$ and $(-3 \pm i)/2$. Because $S < 0$, the validity condition [3, (33)] is violated (cf. (4.25)), and the integral of the first kind would have to be split in two parts before using [3, (34)].) In each of the formulas (2.36) to (2.48) the integral on the left side, defined by (2.35), was integrated numerically by the SLATEC code QNG. On the right side the quantities R_F, G, H, H_0 were calculated by using the codes for R-functions in the Supplements to [4, 5], and the remaining calculations were done with a hand calculator. For each of the 13 cases the values obtained for the two sides agreed to better than one part in a million (better than the claimed accuracy of QNG).

Some intermediate values are

$$
\begin{aligned}
M^2 &= 0.36362947, & R_F(M^2, L_-^2, L_+^2) &= 0.54784092, \\
L_-^2 &= 0.53423014, & R_D(M^2, L_-^2, L_+^2) &= 0.042910488, \\
L_+^2 &= 24.673029, & R_J(M^2, L_-^2, L_+^2, \Omega^2) &= 0.048599080, \\
\Omega^2 &= 21.199185, & R_J(M^2, L_-^2, L_+^2, \Omega_0^2) &= 0.12739513, \\
\Omega_0^2 &= 6.1236295, & R_C(a^2, b^2) &= 0.0098889795, \\
a^2 &= 11237.193, & R_C(a_0^2, b_0^2) &= 0.050085175, \\
b^2 &= 9741.4746, & R_C(T^2, V^2) &= 0.58372845, \\
a_0^2 &= 844.71933, & R_C(T_0^2, V_0^2) &= 0.94657139, \\
b_0^2 &= 247.52253, & G &= 10.495586, \\
T^2 &= 10.288757, & H &= 0.049905556, \\
V^2 &= 1.1362990, & H_0 &= -1.8557835, \\
T_0^2 &= 1.1328716, & A(-1, 1, 1, -1) &= 1.5731367, \\
V_0^2 &= 1.1077327, & A(1, 1, 1, 1) &= -0.49594737, \\
X &= -1.1571677, & A(-1, 1, 1, -1, -2) &= 6.2622360, \\
X_0 &= -0.093427949, & A(1, 1, 1, 1, -2) &= 14.845682.
\end{aligned}
$$

As a test of Cauchy principal values, the three integrals with $p_5 = -2$, viz. (2.39), (2.46), and (2.47), were checked numerically with the same values of x, y, f_i, g_i, and h_i as before but with $a_5 + b_5 t = t$, so that each integrand has a simple pole in the open interval of integration. In each case the Cauchy principal value of the left side was computed by the SLATEC code QAWC, and the right side was calculated as before. Cauchy principal values are not required for either R_J or R_C, as one can see from (4.6) and (2.19), since $V^2 > 0$ whether $\xi_5\eta_5$ is positive or negative. For each of the three cases the values obtained for the two sides agreed to better than one part in a million, even though the SLATEC code issued a warning about impairment of accuracy by roundoff error in the case of (2.47).

APPENDIX A. R_J WITH A RESTRICTED COMPLEX PARAMETER

When the fourth variable of R_J is complex but has a special relation (see (A.2)) to the first three variables, which are real, transformation (A.8) leads to R_J with four real variables. To derive this, we start from the inverse Landen transformation [7, (8.5), (5.7), (7.2)],

$$\text{(A.1)}\quad \begin{aligned} 2(w_+^2 - \alpha^2)R_J(z_-^2, z_+^2, \alpha^2, w_+^2) = (w_+^2 - w_-^2)R_J(x^2, y^2, z^2, w^2) \\ + 3R_F(x^2, y^2, z^2) - 3R_C(z^2, w^2), \end{aligned}$$

$$\begin{gathered} y + x = 2\alpha, \quad y - x = (2/\alpha)[(\alpha^2 - z_+^2)(\alpha^2 - z_-^2)]^{1/2}, \quad z = z_+ z_-/\alpha, \\ w = w_+ w_-/\alpha, \qquad (\alpha^2 - w_+^2)(\alpha^2 - w_-^2) = (\alpha^2 - z_+^2)(\alpha^2 - z_-^2). \end{gathered}$$

We are concerned with the case in which w_+^2 is not real, $\alpha > z_+ > 0$, and $-z_+ \le z_- \le z_+$. (We exclude the degenerate case $\alpha = z_+$, in which R_J is elementary, $w_-^2 = \alpha^2$, and $w^2 = w_+^2$, whence w^2 is complex.) The last equation in (A.1) defines w_-^2 and shows, since the right side is positive, that $\alpha^2 - w_+^2$ and $\alpha^2 - w_-^2$ have equal and opposite complex phases. If they have also the same absolute value, i.e., if

$$\text{(A.2)}\quad |\alpha^2 - w_+^2|^2 = (\alpha^2 - z_+^2)(\alpha^2 - z_-^2),$$

then w_-^2 is the complex conjugate of w_+^2, and hence $w^2 > 0$. Since w cannot vanish, we may choose $w > 0$, whence w_- is the complex conjugate of w_+.

From (A.1) and (A.2) we see that $y > x > 0$ and

$$\text{(A.3)}\quad \begin{aligned} &xy + z^2 = z_+^2 + z_-^2, && xy + w^2 = w_+^2 + w_-^2, \\ &(x+y)z = 2z_+z_-, && (x+y)w = 2w_+w_-, \\ &(x \pm z)(y \pm z) = (z_+ \pm z_-)^2, && (x \pm w)(y \pm w) = (w_+ \pm w_-)^2, \\ &w^2 - z^2 = w_+^2 + w_-^2 - z_+^2 - z_-^2. \end{aligned}$$

We find also that

$$\text{(A.4)}\quad \begin{aligned} x^2 - z^2 &= [(\alpha^2 - z_-^2)^{1/2} - (\alpha^2 - z_+^2)^{1/2}]^2, \\ y^2 - z^2 &= [(\alpha^2 - z_-^2)^{1/2} + (\alpha^2 - z_+^2)^{1/2}]^2, \\ x^2 - w^2 &= [(\alpha^2 - w_-^2)^{1/2} - (\alpha^2 - w_+^2)^{1/2}]^2, \\ y^2 - w^2 &= [(\alpha^2 - w_-^2)^{1/2} + (\alpha^2 - w_+^2)^{1/2}]^2. \end{aligned}$$

Since $(x - w)(y - w) = (w_+ - w_-)^2 < 0$ and $x^2 - z^2 \ge 0$, we have

$$\text{(A.5)}\quad y > w > x > 0 \quad \text{and} \quad -x \le z \le x.$$

If $z_- < 0$, then $z < 0$, and the R-functions in (A.1) do not take the principal values represented by (1.3) to (1.5). A remedy is provided by the duplication theorem [7, (6.1)(8.7)]:

$$\text{(A.6)}\quad \begin{aligned} R_F(x^2, y^2, z^2) &= 2R_F(x^2 + \lambda, y^2 + \lambda, z^2 + \lambda), \\ R_J(x^2, y^2, z^2, w^2) &= 2R_J(x^2 + \lambda, y^2 + \lambda, z^2 + \lambda, w^2 + \lambda) \\ &\quad + 3R_C(a^2, b^2), \end{aligned}$$

with

$$\lambda = xy + xz + yz, \qquad a = w^2(x+y+z) + xyz,$$
$$b = w(w^2+\lambda), \qquad b \pm a = (w \pm x)(w \pm y)(w \pm z).$$

It follows from (A.5) that $b - a < 0$. Since $z^2 + \lambda = (z+x)(z+y)$ is a product of nonnegative factors, and $w^2 + \lambda > z^2 + \lambda$ by (A.5), we have

$$a > b > 0. \tag{A.7}$$

Finally, we combine (A.6) and (A.1):

Theorem. *If* (A.2) *holds, let* w_- *be the complex conjugate of* w_+. *Then*

$$\begin{aligned} &2(w_+^2 - \alpha^2)R_J(z_-^2, z_+^2, \alpha^2, w_+^2) \\ &\quad = (w_+^2 - w_-^2)[2R_J(x^2+\lambda, y^2+\lambda, z^2+\lambda, w^2+\lambda) + 3R_C(a^2, b^2)] \\ &\qquad + 6R_F(x^2+\lambda, y^2+\lambda, z^2+\lambda) - 3R_C(z^2, w^2), \end{aligned} \tag{A.8}$$

where

$$\begin{aligned} &\alpha > z_+ > 0, \quad -z_+ \le z_- \le z_+, \quad \mathrm{Im}(w_+^2) \ne 0, \\ &z = z_+ z_-/\alpha, \quad w = w_+ w_-/\alpha, \quad z^2 + \lambda = (z_+ + z_-)^2, \\ &x^2 + \lambda = (z_+ + z_-)^2 + [(\alpha^2 - z_-^2)^{1/2} - (\alpha^2 - z_+^2)^{1/2}]^2, \\ &y^2 + \lambda = (z_+ + z_-)^2 + [(\alpha^2 - z_-^2)^{1/2} + (\alpha^2 - z_+^2)^{1/2}]^2, \\ &w^2 + \lambda = w_+^2 + w_-^2 + 2z_+z_-, \qquad a = [(w_+^2 + w_-^2)z_+z_- + 2w_+^2w_-^2]/\alpha, \\ &b = w_+w_-(w_+^2 + w_-^2 + 2z_+z_-)/\alpha, \\ &b \pm a = (w_+ \pm w_-)^2(w_+w_- \pm z_+z_-)/\alpha, \\ &b^2 - a^2 = (w_+^2 - w_-^2)^2(w^2 - z^2) = (w_+^2 - w_-^2)^2(w_+^2 + w_-^2 - z_+^2 - z_-^2). \end{aligned} \tag{A.9}$$

Note that $z^2 + \lambda$ is the square of a nonnegative quantity, even if $z < 0$. The first term on the right side of (A.8) is pure imaginary while the second and third terms are real. If $z < 0$, the third term is not represented by (1.5) until it is rewritten by the duplication theorem as $-6R_C((z+w)^2, 2w(z+w))$; alternatively, it can be expressed in terms of an arctangent taken in the second quadrant rather than the fourth. Neither procedure is needed in this paper.

Appendix B. Real and imaginary parts of R_C

In §4 we need to separate the real and imaginary parts of R_C when its two variables are complex but differ by a real number. (If the difference is not real, it can be made real by using the homogeneity of R_C.)

Lemma. *Let* x, y, c *be real,* $z = x + iy$, $r^2 = x^2 + y^2 > 0$, *and* $z^2 + c \ne 0$. *If* $|c| \le r^2$, *then*

$$\begin{aligned} R_C(z^2, z^2 + c) &= xR_C((r^2 - c)^2, |z^2 + c|^2) \\ &\quad - iyR_C((r^2 + c)^2, |z^2 + c|^2), \end{aligned} \tag{B.1}$$

where $R_C(z^2, z^2 + c)$ *denotes the branch that is continuous in* c *and takes the value* $1/z$ *when* $c = 0$.

Proof. Let $|c| = a^2 \le r^2$. Then (B.1) reduces by [2, (6.9-15), (6.9-16)] to the correct equations

$$\log\frac{z+a}{z-a} = \log\left|\frac{z+a}{z-a}\right| - i\arctan\frac{2ay}{r^2-a^2} \quad \text{if } c = -a^2,$$
$$\arctan\frac{a}{z} = \frac{1}{2}\arctan\frac{2ax}{r^2-a^2} - \frac{i}{4}\log\frac{r^2+2ay+a^2}{r^2-2ay+a^2} \quad \text{if } c = a^2.$$

On each right-hand side, the logarithm is taken real and the arctangent is taken in the first or fourth quadrant to get the principal value of the left side.

In the excluded case when $|c| > r^2$, the arctangent must be taken in the third or second quadrant, and the corresponding R_C in (B.1) has the square of a negative number as its first argument. This can be replaced by the square of a positive number by using the duplication theorem [7, (3.7)], and c can then have any real value provided $z^2 + c \neq 0$. The result is given here although it is not needed in the present paper; it provides a way of computing R_C with complex arguments.

Theorem. *Let* x, y, c *be real,* $z = x + iy$, $r^2 = x^2 + y^2 > 0$, *and* $s = |z^2 + c| > 0$. *Then*

$$\begin{aligned} &R_C(z^2, z^2+c) = xR_C(\sigma_-^2, s\sigma_-) - iyR_C(\sigma_+^2, s\sigma_+), \\ \text{(B.2)} \qquad &s^2 = |z^2+c|^2 = (r^2-c)^2 + 4cx^2 = (r^2+c)^2 - 4cy^2, \\ &\sigma_\pm = (r^2 \pm c + s)/2 > 0. \end{aligned}$$

In the first equation, $R_C(z^2, z^2+c)$ denotes the branch that is continuous in c and takes the value $1/z$ when $c = 0$. In the exceptional cases where z^2 is real and $c/z^2 < -1$, it denotes the Cauchy principal value of $(1/z)R_C(1, 1+c/z^2)$. On the right side of the first equation, each R_C denotes the principal branch represented by (1.5).

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